A report to reading project on

Introduction to Brownian Bridges

A Mathematical Exploration of Brownian Motion and Stochastic Processes

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Abstract

Brownian motion is one of the fundamental mathematical models for continuous random motion. Although Brownian motion starts from a fixed point, its position at a future time is random. This project studies what happens when this randomness is constrained by fixing the endpoint of the process. The resulting process is known as a Brownian bridge.

The project begins with the probability concepts required to understand continuous-time stochastic processes and then introduces Brownian motion and some of its fundamental properties. A standard Brownian bridge from 0 to 0 over a fixed time interval $[0, T]$ is constructed from Brownian motion using

$$B_t = W_t - \frac{t}{T} W_T.$$

The mean, variance, and covariance of the Brownian bridge are derived and interpreted. In particular, the variance

$$\mathrm{Var}(B_t) = \frac{t(T-t)}{T}$$

shows that the uncertainty is zero at both endpoints and greatest in the interior of the interval.

The project also discusses the stochastic differential equation satisfied by a Brownian bridge,

$$dB_t = -\frac{B_t}{T-t}\, dt + dW_t,$$

which gives an intuitive interpretation of the bridge as a random process with a restoring drift that pulls it toward its prescribed endpoint. Finally, applications are discussed to illustrate how Brownian bridges can be used to model conditioned random processes.

Keywords: Brownian motion, Brownian bridge, stochastic process, Gaussian process, stochastic differential equation

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Chapter 1

Introduction

Mathematical models are often used to describe systems whose behaviour changes over time. In some situations, the future behaviour of a system can be predicted exactly once its initial conditions are known. Such models are called deterministic. In other situations, randomness plays an essential role, and the future cannot be predicted exactly. These models are described using probability and stochastic processes.

A stochastic process is a mathematical model for a quantity that evolves randomly with time. Such processes appear in many areas of mathematics and science, including physics, finance, biology, engineering and statistics.

One of the most important examples is *Brownian motion*. Brownian motion provides a mathematical model for continuous random movement. It has a simple definition, but its sample paths have remarkable properties: they are continuous but, with probability one, nowhere differentiable.

This creates difficulties for ordinary differential calculus. To study Brownian motion mathematically, a different framework called *stochastic calculus* is used. It includes the Itô integral and Itô's lemma.

The main purpose of this project is not to develop stochastic calculus in complete generality, but to use its basic ideas to understand a particularly interesting stochastic process: the *Brownian bridge*.

1.1 Research Question

The central question of this project is:

> *How can Brownian motion be conditioned to reach a prescribed endpoint, and what mathematical properties does the resulting Brownian bridge possess?*

1.2 Objectives

The objectives of the project are:

1. To introduce the probability concepts required to study stochastic processes.
2. To understand the definition and fundamental properties of Brownian motion.
3. To construct a Brownian bridge from Brownian motion.
4. To derive the mean, variance and covariance of a Brownian bridge.
5. To understand the stochastic differential equation of a Brownian bridge.
6. To simulate Brownian bridges and interpret their behaviour.
7. To discuss possible applications of Brownian bridges.

1.3 Organisation of the Report

Chapter 2 introduces the basic probability concepts required later. Chapter 3 develops the idea of Brownian motion. Chapter 4 introduces Brownian bridges and explains their construction. Chapter 5 derives their main mathematical properties and studies their stochastic differential equation. Chapter 6 discusses simulation, while Chapter 7 describes applications and interpretations. The final chapter summarises the main ideas.

Chapter 2

Probability and Stochastic Foundations

Probability provides the mathematical framework for describing uncertainty and randomness. It is the foundation for stochastic processes, stochastic calculus, and mathematical models used in finance. This chapter introduces the basic concepts required for the study of continuous-time stochastic models.

2.1 Random Experiments and Probability

A *random experiment* is an experiment whose outcome cannot be predicted with certainty in advance, although the set of possible outcomes is known.

Definition 2.1 (Sample Space)**.** *The* sample space*, denoted by* Ω*, is the set of all possible outcomes of a random experiment.*

An individual outcome is denoted by $\omega \in \Omega$. Events are collections of outcomes.

Definition 2.2 (Event)**.** *An* event *is a subset* $A \subseteq \Omega$ *of the sample space.*

The collection of events to which probabilities can be assigned is described by a σ-algebra.

Definition 2.3 (σ-algebra)**.** *A collection* $\mathcal{F}$ *of subsets of* Ω *is a* σ-algebra *if*

1. $\Omega \in \mathcal{F}$;
2. *if* $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$;
3. *if* $A_1, A_2, \ldots \in \mathcal{F}$, *then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 2.4 (Probability Measure). *A* probability measure *is a function*

$$P : \mathcal{F} \to [0, 1]$$

satisfying the Kolmogorov axioms:

1. $P(\Omega) = 1$;

   "'

2. *if* $A \in \mathcal{F}$, *then*
$$P(A) \geq 0;$$

3. *for pairwise disjoint events* $A_1, A_2, \ldots$,
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

   "'

The triple

$$(\Omega, \mathcal{F}, P)$$

is called a *probability space*.
Some useful consequences are

$$P(A^c) = 1 - P(A),$$

and

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

2.2 Conditional Probability and Independence

Conditional probability measures the probability of an event when another event is known to have occurred.

Definition 2.5 (Conditional Probability). *For events* $A, B \in \mathcal{F}$ *with* $P(B) > 0$, *the conditional probability of* A *given* B *is*

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}.}$$

The multiplication rule follows immediately:

$$P(A \cap B) = P(A \mid B)P(B).$$

More generally,

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

Definition 2.6 (Independence)**.** *Two events A and B are* independent *if*

$$P(A \cap B) = P(A)P(B).$$

Equivalently,

$$P(A \mid B) = P(A)$$

whenever $P(B) > 0$.

Random variables are independent if the occurrence of one does not provide information about the distribution of the other.

Bayes' theorem follows from the definition of conditional probability:

$$\boxed{P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.}$$

If $A_1, \ldots, A_n$ is a partition of Ω, then

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i),$$

and hence

$$P(A_j \mid B) = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)}.$$

2.3 Random Variables

Definition 2.7 (Random Variable)**.** *A* random variable *is a measurable function*

$$X : \Omega \to \mathbb{R}.$$

It assigns a numerical value to each outcome of a random experiment.

The distribution of a random variable describes how probability is assigned to its possible values.

Random variables may be *discrete*, *continuous*, or more generally *mixed*.

2.3.1 Probability Mass Function

If X is discrete, its probability mass function (PMF) is

$$p_X(x) = P(X = x).$$

It satisfies

$$p_X(x) \geq 0, \qquad \sum_x p_X(x) = 1.$$

For a set $A \subseteq \mathbb{R}$,

$$P(X \in A) = \sum_{x \in A} p_X(x).$$

2.3.2 Probability Density Function

If X is continuous and has density f_X, then

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx,$$

where

$$f_X(x) \geq 0, \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

For a continuous random variable,

$$P(X = x) = 0$$

for every individual x.

2.3.3 Cumulative Distribution Function

Definition 2.8 (CDF)**.** *The* cumulative distribution function *of a random variable X is*

$$\boxed{F_X(x) = P(X \leq x).}$$

Every CDF is non-decreasing and satisfies

$$\lim_{x \to -\infty} F_X(x) = 0, \qquad \lim_{x \to \infty} F_X(x) = 1.$$

For a continuous random variable with density f_X,

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du,$$

and, wherever differentiable,

$$f_X(x) = F'_X(x).$$

2.4 Expectation and Variance

Definition 2.9 (Expectation)**.** *For a discrete random variable X,*

$$\mathbb{E}[X] = \sum_x x p_X(x),$$

provided the sum is absolutely convergent. For a continuous random variable,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

More generally, for a suitable function g,

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) p_X(x), & X \text{ discrete,} \\ \int_{-\infty}^{\infty} g(x) f_X(x)\, dx, & X \text{ continuous.} \end{cases}$$

Expectation is linear:

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

Definition 2.10 (Variance)**.** *The* variance *of X is*

$$\boxed{\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].}$$

Equivalently,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

The standard deviation is

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

2.5 Covariance and Correlation

Definition 2.11 (Covariance)**.** *For random variables X and Y, their covariance is*

$$\boxed{\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right].}$$

Equivalently,

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Definition 2.12 (Correlation)**.** *The correlation coefficient between X and Y is*

$$\boxed{\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y},}$$

provided $\sigma_X, \sigma_Y > 0$.

The correlation satisfies

$$-1 \le \rho_{X,Y} \le 1.$$

If X and Y are independent and the relevant expectations exist, then

$$\mathrm{Cov}(X,Y) = 0.$$

The converse need not hold in general.

2.6 Joint and Conditional Distributions

For two random variables X and Y, their joint CDF is

$$F_{X,Y}(x,y) = P(X \le x, Y \le y).$$

If a joint density exists, it is denoted by

$$f_{X,Y}(x,y).$$

The marginal density of X is obtained by integrating out Y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

When $f_Y(y) > 0$, the conditional density of X given $Y = y$ is

$$\boxed{f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.}$$

The random variables X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

where the densities exist.

2.7 Normal Distribution

The normal distribution is one of the most important probability distributions in statistics and mathematical finance.

Definition 2.13 (Normal Distribution)**.** *A random variable X has a normal distribution with mean μ and variance σ^2, written*

$$X \sim N(\mu, \sigma^2),$$

if its density is

$$\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad \sigma > 0.}$$

For $X \sim N(\mu, \sigma^2)$,

$$\mathbb{E}[X] = \mu, \qquad \mathrm{Var}(X) = \sigma^2.$$

The standard normal random variable is

$$Z \sim N(0,1).$$

Any normal random variable can be standardized using

$$Z = \frac{X-\mu}{\sigma}.$$

An important property is that a linear combination of jointly normally distributed random variables is again normally distributed.

2.8 Stochastic Processes

The concepts above describe individual random variables. To model quantities that evolve over time, we use stochastic processes.

Definition 2.14 (Stochastic Process)**.** *A* stochastic process *is a collection of random variables*

$$\{X_t : t \in T\}$$

defined on a common probability space, where T is an index set representing time.

For each fixed t, X_t is a random variable. For a fixed outcome $\omega \in \Omega$, the function

$$t \mapsto X_t(\omega)$$

is called a *sample path* or *realization.*
Stochastic processes can be indexed by discrete time,

$$t = 0, 1, 2, \ldots,$$

or continuous time,

$$t \geq 0.$$

A stochastic process is therefore a natural mathematical model for a quantity whose value changes randomly through time. For example, an asset price can be represented by

$$\{S_t : t \geq 0\}.$$

2.8.1 Gaussian Processes

Definition 2.15 (Gaussian Process)**.** *A stochastic process $X_t : t \in T$ is a* Gaussian process *if for every finite collection of times*

$$t_1, \ldots, t_n,$$

the random vector

$$(X_{t_1}, \ldots, X_{t_n})$$

has a multivariate normal distribution.

A Gaussian process is completely characterized by its mean function

$$m(t) = \mathbb{E}[X_t]$$

and covariance function

$$C(s, t) = \mathrm{Cov}(X_s, X_t).$$

2.8.2 Markov Property

A process is Markov if, conditional on its present state, its future is independent of its past.

Informally,

$$\text{future} \perp\!\!\!\perp \text{past} \mid \text{present}.$$

More precisely, for $s < t$ and suitable events A,

$$P(X_t \in A \mid \mathcal{F}_s) = P(X_t \in A \mid X_s).$$

The Markov property is particularly important in continuous-time stochastic models and will be used when studying Brownian motion.

2.9 From Probability to Stochastic Calculus

A stochastic process describes the evolution of a random quantity through time. In many applications, however, we need to describe not only the process itself but also its rate of change and accumulated random effects.

For a deterministic differentiable function $x(t)$, ordinary calculus uses

$$dx_t = x'(t)\,dt$$

and

$$\int_0^T x'(t)\,dt = x(T) - x(0).$$

For random processes with sufficiently irregular paths, ordinary differentiation and integration may no longer be appropriate. In particular, Brownian motion provides a continuous random process whose sample paths are almost surely nowhere differentiable.

This motivates the development of *stochastic calculus*, in which integration with respect to random processes and stochastic differential equations are defined carefully.

The next chapter introduces Brownian motion and develops the basic tools of stochastic calculus, including quadratic variation, the Itô integral, and Itô's lemma.

Chapter 3

Introduction to Stochastic Calculus

Stochastic calculus provides the mathematical framework for studying systems that evolve over time under the influence of randomness. While ordinary calculus deals with deterministic functions, stochastic calculus deals with random processes whose values change unpredictably with time.

The central object in this chapter is Brownian motion, which provides a basic model for continuous-time random fluctuations. We introduce stochastic processes, Brownian motion, stochastic differentials, quadratic variation, the Itô integral, and Itô's lemma.

3.1 Stochastic Processes

Definition 3.1 (Stochastic Process)**.** *A* stochastic process *is a collection of random variables*

$$\{X_t : t \in T\}$$

defined on a common probability space $(\Omega, \mathcal{F}, P)$*, where* T *is an index set representing time.*

For each fixed t, X_t is a random variable. For a fixed $\omega \in \Omega$, the function

$$t \longmapsto X_t(\omega)$$

is called a *sample path* or *realization* of the process.

Thus, a stochastic process can be viewed in two ways:

$$X_t : \Omega \to \mathbb{R}$$

for fixed t, or as a function of time

$$t \mapsto X_t(\omega)$$

for fixed ω.

Stochastic processes may be indexed by discrete or continuous time. In this chapter we primarily consider continuous-time processes,

$$\{X_t : t \geq 0\}.$$

For example, if S_t denotes the price of an asset at time t, then

$$\{S_t : t \geq 0\}$$

can be modelled as a stochastic process.

Definition 3.2 (Filtration)**.** *A* filtration $\mathcal{F}_t : t \geq 0$ *is an increasing family of σ-algebras satisfying*

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \qquad 0 \leq s \leq t.$$

It represents the information available up to time t.

Definition 3.3 (Adapted Process)**.** *A process X_t is* adapted *to $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$.*

Thus, an adapted process does not depend on information from the future. This non-anticipation property is important in stochastic integration.

3.2 Brownian Motion

Brownian motion, also called the Wiener process, is the fundamental continuous-time stochastic process used in stochastic calculus.

Definition 3.4 (Standard Brownian Motion)**.** *A stochastic process $W_t : t \geq 0$ is a* standard Brownian motion *if:*

1. *$W_0 = 0$ almost surely;*

   “‘

2. *for*

   $$0 \leq t_0 < t_1 < \cdots < t_n,$$

   the increments

   $$W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$$

   are independent;

3. for $0 \leq s < t$*,*

$$W_t - W_s \sim N(0, t-s);$$

4. with probability one, the sample paths $t \mapsto W_t$ *are continuous.* "'

The increment property gives

$$\mathbb{E}[W_t - W_s] = 0$$

and

$$\mathrm{Var}(W_t - W_s) = t - s.$$

In particular,

$$W_t \sim N(0, t),$$

so

$$\mathbb{E}[W_t] = 0, \qquad \mathrm{Var}(W_t) = t.$$

Brownian motion has both independent and stationary increments. Its covariance function is

$$\mathrm{Cov}(W_s, W_t) = \min(s, t).$$

Since every finite collection

$$(W_{t_1}, \ldots, W_{t_n})$$

has a multivariate normal distribution, Brownian motion is a Gaussian process. It also has the Markov property: conditional on its present value, its future evolution is independent of its past.

3.2.1 Path Properties

Brownian motion has continuous paths, but they are highly irregular. In particular, Brownian motion is almost surely nowhere differentiable:

$$\frac{dW_t}{dt}$$

does not exist in the ordinary sense with probability one.

Thus, although dW_t is used extensively in stochastic calculus, it cannot be interpreted simply as

$$dW_t = W_t' \, dt.$$

Instead, dW_t represents a stochastic increment whose mathematical meaning is defined through limits of sums and stochastic integration.

A useful heuristic is

$$W_{t+\Delta t} - W_t \sim N(0, \Delta t),$$

so that

$$\Delta W = O(\sqrt{\Delta t}).$$

This scaling is responsible for the distinctive rules of stochastic calculus introduced below.

3.3 Generalized Wiener Processes

A simple extension of Brownian motion is the generalized Wiener process.

Definition 3.5 (Generalized Wiener Process)**.** *A process X_t satisfying*

$$dX_t = a \, dt + b \, dW_t,$$

where a, b are constants, is called a generalized Wiener process*.*

In integral form,

$$X_t = X_0 + at + bW_t.$$

Hence

$$X_t \sim N(X_0 + at, b^2 t),$$

and therefore

$$\mathbb{E}[X_t] = X_0 + at, \qquad \mathrm{Var}(X_t) = b^2 t.$$

The coefficient a is called the *drift*, while b is the *diffusion coefficient* or *volatility.*

More generally, an Itô process has the form

$$dX_t = a(t, X_t) \, dt + b(t, X_t) \, dW_t.$$

The term involving dt represents the systematic component of the change, whereas the term involving dW_t represents the random component.

3.4 Why Ordinary Calculus Fails

For a differentiable deterministic function $x(t)$,

$$dx_t = x'(t)\,dt.$$

One might therefore expect a similar relation for Brownian motion. However, Brownian paths are almost surely nowhere differentiable.

The fundamental difference is revealed by considering the size of increments. For an ordinary differentiable function,

$$\Delta x = O(\Delta t),$$

and hence

$$(\Delta x)^2 = O((\Delta t)^2).$$

For Brownian motion,

$$\Delta W = O(\sqrt{\Delta t}),$$

and consequently

$$(\Delta W)^2 = O(\Delta t).$$

Thus, second-order terms involving Brownian increments do not disappear in the limit. This leads to the concept of quadratic variation.

3.5 Quadratic Variation

Consider a partition of $[0, t]$,

$$0 = t_0 < t_1 < \cdots < t_n = t,$$

and define its mesh by

$$\|\Pi\| = \max_{0 \le i < n} (t_{i+1} - t_i).$$

Definition 3.6 (Quadratic Variation). *The* quadratic variation *of a process X over $[0, t]$, when the limit exists, is*

$$[X]_t = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \left(X_{t_{i+1}} - X_{t_i} \right)^2 .$$

For Brownian motion,

$$\boxed{[W]_t = t.}$$

In stochastic differential notation, this is expressed through the fundamental Itô rules

$$\boxed{(dW_t)^2 = dt,}$$

$$\boxed{dt\, dW_t = 0,}$$

and

$$\boxed{(dt)^2 = 0.}$$

These are operational rules describing the limiting behaviour of the corresponding terms; they are not ordinary algebraic identities.

For a generalized Wiener process

$$dX_t = a\, dt + b\, dW_t,$$

the quadratic variation is

$$[X]_t = b^2 t.$$

3.6 The Itô Integral

Ordinary calculus defines integrals with respect to time, such as

$$\int_0^T f(t)\, dt.$$

Stochastic calculus also requires integration with respect to Brownian motion.

Definition 3.7 (Itô Integral). *Let H_t be an adapted process satisfying*

$$\mathbb{E}\left[\int_0^T H_t^2\, dt\right] < \infty.$$

The Itô integral *is defined as the limit*

$$\int_0^T H_t\, dW_t = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} H_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right),$$

under the appropriate mode of convergence.

The use of the left endpoint t_i is essential. The value H_{t_i} depends only on information available at time t_i, whereas

$$W_{t_{i+1}} - W_{t_i}$$

is the future increment. Thus, the Itô integral is based on a non-anticipating integrand. For example,

$$\int_0^T dW_t = W_T.$$

A fundamental property is that the Itô integral has zero expectation:

$$\boxed{\mathbb{E}\left[\int_0^T H_t \, dW_t\right] = 0.}$$

It also satisfies the Itô isometry.

Theorem 3.1 (Itô Isometry)**.** *For a suitable adapted process H_t,*

$$\boxed{\mathbb{E}\left[\left(\int_0^T H_t \, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T H_t^2 \, dt\right].}$$

The Itô isometry is fundamental for establishing the existence and properties of stochastic integrals.

3.6.1 Example: An Itô Integral

Consider

$$\int_0^T W_t \, dW_t.$$

Unlike an ordinary integral, this cannot simply be written as

$$\frac{1}{2} W_T^2.$$

Applying Itô's lemma to $f(x) = x^2$ gives

$$d(W_t^2) = 2W_t \, dW_t + dt.$$

Integrating from 0 to T,

$$W_T^2 = 2\int_0^T W_t \, dW_t + T.$$

Hence

$$\boxed{\int_0^T W_t\,dW_t = \frac{1}{2}(W_T^2 - T).}$$

The extra $-T$ term is a direct consequence of the quadratic variation of Brownian motion.

3.7 Itô's Lemma

Itô's lemma is the stochastic analogue of the chain rule. Let

$$X_t$$

be an Itô process satisfying

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t.$$

If $f(t, x)$ is sufficiently smooth, specifically $f \in C^{1,2}$, then Itô's lemma gives the differential of

$$f(t, X_t).$$

Theorem 3.2 (Itô's Lemma)**.** *Let*

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t.$$

For $f \in C^{1,2}$*,*

$$\boxed{\begin{aligned} df(t, X_t) = &\left[f_t(t, X_t) + a(t, X_t) f_x(t, X_t) + \frac{1}{2} b(t, X_t)^2 f_{xx}(t, X_t)\right] dt \\ &+ b(t, X_t) f_x(t, X_t)\,dW_t. \end{aligned}}$$

The additional term

$$\frac{1}{2} b(t, X_t)^2 f_{xx}(t, X_t)\,dt$$

is the main difference between Itô's lemma and the ordinary chain rule. It arises from

$$(dW_t)^2 = dt.$$

3.7.1 Derivation by Taylor Expansion

Formally, consider a second-order Taylor expansion:

$$df = f_t\,dt + f_x\,dX_t + \frac{1}{2} f_{xx}(dX_t)^2.$$

Since

$$dX_t = a\,dt + b\,dW_t,$$

we have

$$\begin{aligned}(dX_t)^2 &= (a\,dt + b\,dW_t)^2 \\ &= a^2(dt)^2 + 2ab\,dt\,dW_t + b^2(dW_t)^2.\end{aligned}$$

Using

$$(dt)^2 = 0, \qquad dt\,dW_t = 0, \qquad (dW_t)^2 = dt,$$

we obtain

$$(dX_t)^2 = b^2\,dt.$$

Substitution gives

$$df = \left(f_t + af_x + \frac{1}{2}b^2 f_{xx}\right) dt + bf_x\,dW_t.$$

This provides the intuition behind Itô's lemma.

3.7.2 Special Case

Suppose

$$dX_t = dt + dW_t.$$

Then

$$a = 1, \qquad b = 1,$$

and Itô's lemma becomes

$$\boxed{df(t, X_t) = \left(f_t + f_x + \frac{1}{2} f_{xx}\right) dt + f_x\,dW_t.}$$

As another important example, if

$$X_t = W_t$$

and

$$f(x) = x^2,$$

then

$$f_x = 2x, \qquad f_{xx} = 2,$$

and therefore

$$\boxed{d(W_t^2) = 2W_t\, dW_t + dt.}$$

Chapter 4

Brownian Bridges

4.1 Motivation

Brownian motion begins at a fixed point but has a random position at a future time. Suppose instead that we know not only where the process starts, but also where it must end.

For example, suppose $W_0 = 0$ but we require the process to return to zero at a fixed time T, i.e., $W_T = 0$.

This leads to the idea of a Brownian bridge.

4.2 Definition of a Standard Brownian Bridge

Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. A standard Brownian bridge on $[0, T]$ is defined by

$$\boxed{B_t = W_t - \frac{t}{T}W_T, \qquad 0 \leq t \leq T.}$$

This construction removes the part of the Brownian motion associated with its random endpoint and forces the process to return to zero.

At the initial time,

$$B_0 = W_0 - \frac{0}{T}W_T = 0.$$

At the final time,

$$B_T = W_T - \frac{T}{T}W_T = 0.$$

Thus,

$$\boxed{B_0 = B_T = 0.}$$

4.3 Conditional Interpretation

The Brownian bridge can also be interpreted as Brownian motion conditioned on the event

$$W_T = 0.$$

More precisely, the Brownian bridge describes the conditional law of Brownian motion given its value at time T.

This interpretation explains why the process is constrained near the endpoint. As the time t approaches T, there is less time available for the process to reach its prescribed final value.

Properties of the Brownian Bridge

4.3.1 Mean

From the definition

$$B_t = W_t - \frac{t}{T} W_T,$$

and the fact that

$$\mathbb{E}[W_t] = \mathbb{E}[W_T] = 0,$$

we obtain

$$\begin{aligned} \mathbb{E}[B_t] &= \mathbb{E}[W_t] - \frac{t}{T}\mathbb{E}[W_T] \\ &= 0. \end{aligned}$$

Therefore,

$$\boxed{\mathbb{E}[B_t] = 0.}$$

4.3.2 Variance

We calculate

$$\operatorname{Var}(B_t) = \operatorname{Var}\left(W_t - \frac{t}{T} W_T\right).$$

Using

$$\operatorname{Var}(X - aY) = \operatorname{Var}(X) + a^2 \operatorname{Var}(Y) - 2a \operatorname{Cov}(X, Y),$$

we get

$$\mathrm{Var}(B_t) = \mathrm{Var}(W_t) + \frac{t^2}{T^2}\,\mathrm{Var}(W_T) - 2\frac{t}{T}\,\mathrm{Cov}(W_t, W_T).$$

Since

$$\mathrm{Var}(W_t) = t, \qquad \mathrm{Var}(W_T) = T, \qquad \mathrm{Cov}(W_t, W_T) = t,$$

we obtain

$$\begin{aligned}
\mathrm{Var}(B_t) &= t + \frac{t^2}{T} - 2\frac{t^2}{T} \\
&= t - \frac{t^2}{T} \\
&= \boxed{\frac{t(T-t)}{T}}.
\end{aligned}$$

This formula gives an important insight into the behaviour of the bridge.

At $t = 0$,

$$\mathrm{Var}(B_0) = 0,$$

and at $t = T$,

$$\mathrm{Var}(B_T) = 0.$$

The variance is largest at

$$t = \frac{T}{2}.$$

Thus, the bridge has the greatest freedom to fluctuate near the middle of the interval and becomes increasingly constrained near its endpoints.

4.3.3 Covariance

Let $0 \le s, t \le T$. Using

$$B_s = W_s - \frac{s}{T}W_T$$

and

$$B_t = W_t - \frac{t}{T}W_T,$$

we have

$$
\begin{aligned}
\mathrm{Cov}(B_s, B_t) = {} & \mathrm{Cov}(W_s, W_t) - \frac{t}{T}\,\mathrm{Cov}(W_s, W_T) \\
& - \frac{s}{T}\,\mathrm{Cov}(W_T, W_t) + \frac{st}{T^2}\,\mathrm{Var}(W_T).
\end{aligned}
$$

Using

$$\mathrm{Cov}(W_s, W_t) = \min(s, t),$$

together with

$$\mathrm{Cov}(W_s, W_T) = s, \qquad \mathrm{Cov}(W_t, W_T) = t,$$

and

$$\mathrm{Var}(W_T) = T,$$

we obtain

$$
\begin{aligned}
\mathrm{Cov}(B_s, B_t) &= \min(s, t) - \frac{st}{T} - \frac{st}{T} + \frac{st}{T} \\
&= \boxed{\min(s, t) - \frac{st}{T}}.
\end{aligned}
$$

Setting $s = t$ gives the variance obtained above:

$$\mathrm{Var}(B_t) = t - \frac{t^2}{T}.$$

4.3.4 Gaussian Nature

Since Brownian motion is a Gaussian process and B_t is a linear combination of values of Brownian motion, the Brownian bridge is also a Gaussian process.

Consequently, its finite collections of random variables have a joint normal distribution.

This means that the mean and covariance completely determine its finite-dimensional distributions.

4.4 The Stochastic Differential Equation of a Brownian Bridge

The Brownian bridge has a particularly interesting stochastic differential equation:

$$\boxed{dB_t = -\frac{B_t}{T - t}\,dt + dW_t, \qquad 0 \le t < T.}$$

The equation consists of two components.

The term

$$dW_t$$

represents the random fluctuations of Brownian motion.

The term

$$-\frac{B_t}{T-t}\,dt$$

is a deterministic drift depending on the current position of the bridge.

4.4.1 Interpretation of the Drift

Suppose that

$$B_t > 0.$$

Then

$$-\frac{B_t}{T-t} < 0.$$

Thus, the drift acts downward, pulling the process toward zero.

Similarly, if

$$B_t < 0,$$

then

$$-\frac{B_t}{T-t} > 0,$$

so the drift acts upward.

The drift therefore behaves like a restoring force.

Furthermore, as t approaches T,

$$\frac{1}{T-t} \longrightarrow \infty.$$

Thus, the restoring effect becomes stronger as the endpoint approaches. This reflects the fact that the process has less and less time to reach its prescribed endpoint.

4.4.2 An Alternative Representation

The Brownian bridge also has the representation

$$\boxed{B_t = (T-t)\int_0^t \frac{1}{T-s}\,dW_s.}$$

This representation is useful for understanding the bridge within the framework of stochastic integration and can be used to derive its stochastic differential equation.

4.5 General Brownian Bridges

The standard Brownian bridge starts at 0 and ends at 0. We can generalise this to a process starting at a and ending at b.

The Brownian bridge from a at time 0 to b at time T can be represented as

$$\boxed{X_t = a + \frac{t}{T}(b-a) + \left(W_t - \frac{t}{T}W_T\right).}$$

This expression has two components.

The first component,

$$a + \frac{t}{T}(b-a),$$

is the deterministic linear interpolation between a and b.

The second component,

$$W_t - \frac{t}{T}W_T,$$

is the random Brownian bridge fluctuation.

Thus, we can interpret the general bridge as

$$\boxed{\text{linear path} + \text{random fluctuations.}}$$

At $t = 0$,

$$X_0 = a,$$

while at $t = T$,

$$X_T = b.$$

Chapter 5

Applications

Brownian bridges arise naturally whenever a random process is studied under a condition on its endpoint.

5.1 Statistics

Brownian bridges occur in the asymptotic theory of several statistical tests. In particular, they are related to the limiting distributions appearing in goodness-of-fit and empirical distribution problems.

5.2 Probability Theory

Brownian bridges are useful for studying conditioned stochastic processes and provide an important example of how conditioning changes the behaviour of Brownian motion.

5.3 Finance

Stochastic processes are widely used in mathematical finance. Brownian motion is a fundamental component of many continuous-time financial models. Brownian bridges can also be useful when modelling paths subject to known or conditioned values at particular times.

The purpose of this project is not to develop these applications in detail, but they demonstrate that Brownian bridges are more than a purely theoretical construction.

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